Math 144 Discussion 9-29-09

Definition 1 *A* partial ordering *on a set S is a binary relation $\preceq$ on S such that for all $x, y, z \in S$:*

1. $x \preceq x$
2. *if* $x \preceq y$ *and* $y \preceq x$, *then* $x = y$
3. *if* $x \preceq y$ *and* $y \preceq z$, *then* $x \preceq z$

Definition 2 *A* partially ordered set *(or* poset*) is a set S together with some partial ordering $\preceq$ (this will commonly be written as* $(S, \preceq)$

For the following 4 definitions: let S be a subset of a partially ordered set (or poset) L.

Definition 3 Upper Bound - *an element $u \in L$ is an upper bound of S if* $\forall s \in S$, $s \leq u$

Definition 4 Least Upper Bound - *$u \in L$ is the least upper bound (or supremum) of S if it is an upper bound of S and $u \leq v$ for all other upper bounds $v \in L$ of S*

Definition 5 Lower Bound - *an element $u \in L$ is an lower bound of S if* $\forall s \in S$, $u \leq s$

Definition 6 Greatest Lower Bound - *$u \in L$ is the greatest lower bound (or infimum) of S if it is a lower bound of S and $v \leq u$ for all other lower bounds $v \in L$ of S*

Definition 7 *A poset S is said to have the* Least Upper Bound Property *if for E a nonempty subset of S which is bounded above,* $\sup E \in S$.

Remark 8 *A poset which has the Least Upper Bound Property also has the* Greatest Lower Bound Property.

Exercise Let A be the set of all positive rational numbers p such that $p^2 < 2$ and let B be the set of all positive rational numbers p such that $p^2 > 2$. Show that A has no least upper bound in $\mathbb{Q}$ (has no largest number) and that B has no greatest lower bound in $\mathbb{Q}$ (has no smallest number).

Solution What we are going to do here is show that for every $p \in A$ we can find an element $q \in A$ such that $p < q$, and for every $p \in B$ $\exists$ $q \in B$ such that $q < p$.

Associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \qquad (1)$$

Then

$$q^2 - 2 = \frac{2\left(p^2 - 2\right)}{\left(p + 2\right)^2}. \qquad (2)$$

So if $p \in A$ then $p^2 - 2 < 0$, thus (1) shows that $q > p$, and (2) shows that $q^2 < 2$. Thus $q \in A$. Since we can carry on this process indefinitely, A has no largest element.

Now if $p \in B$ then $p^2 - 2 > 0$, thus (1) shows that $0 < q < p$, and (2) shows that $q^2 > 2$. Thus $q \in B$. Again, since we can carry on this process indefinitely, B has no smallest element.

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Remark The purpose of this exercise was to show that $\mathbb{Q}$ has certain gaps, even though it is dense-in-itself (which means that in between any two elements of the set there is another element of the set). The real number system, $\mathbb{R}$, fills these gaps, which is why it is fundamental to analysis.

So we just showed that $\mathbb{Q}$ has a gap because $\nexists$ a rational number whose square is 2. What number would $\mathbb{Q}$ need to contain in order to fill this gap? How does $\mathbb{R}$ fix this?

So we can basically define $\mathbb{R}$ as the combination of $\mathbb{Q}$ and the numbers that fill its gaps. But what are the numbers that fill the gaps in $\mathbb{Q}$ called?

Here is a precise definition of dense-in-itself:

Definition 9 *Let L be a poset. L is said to be* dense-in-itself *if for any two distinct elements $p, q \in L$ with $p < q$ $\exists$ $r \in L$ such that $p < r < q$.*

Example Lets illustrate the above definition with $\mathbb{Q}$. Let $p, q \in \mathbb{Q}$ with $p < q$. Define $r = \frac{p+q}{2}$. Then $r \in \mathbb{Q}$ since the operations of addition and multiplication are closed in $\mathbb{Q}$. Now we have $p < r$ and $r < q$, so $p < r < q$.